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Topological defect with nonzero Hopf invariant in Yang–Mills–Higgs model

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ABSTRACT

We propose a topological defect or instanton solution with nonzero Hopf invariant to the 3+1D non-Abelian gauge theory coupled with scalar fields. This solution, which we call Hopf defect, represents a spacetime event that makes a 2π rotation of vacuum manifold of the monopole. Although the action of this Hopf defect is logarithmically divergent, it may still give relevant contributions in a finite-sized system. Since the Chern–Simons term for the unbroken $U(1)$ gauge field may appear in the low energy effective theory, the Hopf defect may possibly generate a phase factor change for the monopoles.

1. Introduction

Magnetic monopole, although not discovered in Nature, has attracted lots of attention from the theoretical physics community [1]. The first monopole solution was proposed by Dirac [2]. He argued that the existence of the monopole implies quantized electrical charges because of the minimal coupling of the $U(1)$ gauge field in quantum mechanics. However, in his monopole model, the vector potential is singular at some points located on the Dirac's string. This drawback was later overcome in the 't Hooft–Polyakov magnetic monopole solution [3], where the $U(1)$ field is embedded in a non-Abelian gauge field coupled with a scalar field. The Derrick theorem [4] implies that there is no stable soliton solution to the scalar field when the spatial dimension is larger than 1. However, the coupling between the scalar field and the non-Abelian gauge field helps to stabilize this monopole solution.

The magnetic charge of the 't Hooft–Polyakov monopole can be identified as the topological charge, hence it is naturally connected to the classification of the homotopy group $\pi_2(S^2)$. On the contrary, one may simply think that the magnetic charge of the Dirac's monopole has no topological origin. While, it has been shown that the nontrivial $U(1)$ bundle in this model can actually be thought of as a Hopf fabrication [5]. Consequently, the magnetic charge of the Dirac's monopole is connected to the Hopf invariant of another homotopy group $\pi_3(S^2)$ [6]. Therefore the same charge can be attributed to different homotopy groups because of different gauge field realizations of a monopole.

In general, different homotopy groups, such as $\pi_2(S^2)$ and $\pi_3(S^2)$, give rise to different topological solitons. A familiar example is the nonlinear sigma model [7]. In the 2+1D case, this model has a skyrmion-type solution [8] classified by the winding number associated with the group $\pi_2(S^2)$. In a similar way, there is also an instanton-type solution to the same model [9] classified by the nonzero Hopf invariant of the group $\pi_3(S^2)$. This instanton solution represents a 2+1D spacetime event that makes a 2π rotation of the skyrmion. If the Hamiltonian further contains a Chern–Simons term or Hopf term, this instanton can generate fractional statistics for the associated skyrmion. Based on $\pi_3(S^2)$, there is another large class of topological solitons in 3+1D called Hopfion which is a static solution of Faddeev–Skyrme model [10,11]. Hopfion solutions have also been generalized to the Yang–Mills–Higgs model [12,13] and also condensed matter system such as exotic superconductor [14,15] and spinor Bose–Einstein condensates [16].

In contrast to the above-mentioned Hopf solitons, in this paper, we propose a topological defect solution with nonzero Hopf invariant (Hopf defect) to the 3+1D non-Abelian scalar gauge theory. This solution can be thought of as either a soliton-type solution in the spatial part of the 4+1D theory or an instanton-type solution of the 3+1D theory. The relation between the Hopf defect and the non-Abelian magnetic monopole is very similar to that between the skyrmion and the instanton in the nonlinear sigma model discussed above. In the 3+1D case, the solution represents a spacetime event which makes a 2π rotation of vacuum manifold

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of the monopole. In the presence of a Chern–Simons term for the unbroken $U(1)$ gauge field, the monopole will acquire a phase factor under the vacuum space rotation.

The paper is organized as follows. In Section 2, we give a brief review on the Hopf mapping and Hopf invariant. In Section 3, we find out the defect solution with nonzero Hopf invariant. In Sections 4 and 5, we discuss the topological charge and a qualitative physical picture of the solution. We conclude in Section 6.

2. Hopf mapping and Hopf invariant

In this section, we briefly review the Hopf mapping and Hopf invariant, which also helps to identify the topological charge of the Hopf soliton in our later discussions. It is well known that the mapping between two n-dimensional spheres is classified by the n-th homotopy group $\pi_n(S^n) = \mathcal{Z}$. The geometric meaning of the winding number $k \in \mathcal{Z}$ is that when the pre-image point sweeps around the whole sphere, the image point sweeps the whole sphere k times. While, the Hopf mapping is a map between S^3 and S^2, i.e., spheres with different dimensions. Hence it is classified by the homotopy group $\pi_3(S^2)$. The topologically nontrivial Hopf mapping is characterized by the Hopf invariant $\mathcal{H}$. The geometric meaning of the Hopf invariant is not as intuitive as that of the winding number.

Here we give a simple visualization of the Hopf mapping. Under a Hopf mapping, the pre-image of a point on S^2 is a circle in S^3. Hence the pre-images of two different points are two different circles. Under a topologically trivial Hopf mapping, these two circles are not linked. While, under a topologically nontrivial Hopf mapping, these two circles are linked together for one time and form a so-called Hopf link.

Mathematically, we introduce a pair of complex numbers $z_1 = x_1 + ix_2$ and $z_2 = x_3 + ix_4$ to describe $\mathcal{R}^4$, hence the sphere S^3 can be characterized by $|z_1|^2 + |z_2|^2 = 1$. The Hopf mapping $f : S^3 \to S^2$ is given by $y^a = \bar{z}_i \sigma^a_{ij} z_j$ for $a = 1, 2, 3$, where σ^a are Pauli matrices. More explicitly, the Hopf mapping is written as

$$y^1 = 2(x_1x_3 + x_2x_4), \qquad y^2 = 2(x_1x_4 - x_2x_3),$$
$$y^3 = x_1^2 + x_2^2 - x_3^2 - x_4^2, \tag{1}$$

and one can verify that $y^a y^a = |\bar{z}_i z_i|^2 = 1$, then y^a does describe a point on S^2.

The Hopf invariant of the above Hopf mapping (1) can be directly evaluated. Let Ω_2 be the volume 2-form of S^2. Since S^2 is a two-dimensional space, then Ω_2 must be closed, thus we trivially have $d\Omega_2 = 0$. Moreover, Ω_2 must not be exact, otherwise we will have $\Omega = d\alpha$ which implies $\int_{S^2} \Omega_2 = \int_{\partial S^2} \alpha = 0$ by Stokes theorem. This contradicts the fact the volume of S^2 is not zero. The Hopf mapping pulls back the volume 2-form Ω_2 from S^2 to S^3. We define $\omega_2 = f^*\Omega_2$ which is again closed. Since the cohomology of S^3 is trivial, i.e., $H^2(S^3) = 0$, then there is no nontrivial 2-form on S^3. Therefore ω_2 must be exact and can be further written as $\omega_2 = d\omega_1$ where ω_1 is a 1-form on S^3. Finally, the Hopf invariant is defined as

$$\mathcal{H} = \frac{1}{16\pi^2} \int_{S^3} \omega_1 \wedge \omega_2. \tag{2}$$

It easy to verify that $\mathcal{H}$ is invariant under a continuous deformation of the map.

The evaluation of $\mathcal{H}$ can be conveniently performed by using the Cartesian coordinates. The volume 2-form of a unit 2-sphere is given by

$$\Omega_2 = y_1\, dy_2 \wedge dy_3 - y_2\, dy_1 \wedge dy_3 + y_3\, dy_1 \wedge dy_2. \tag{3}$$

Inserting the Hopf mapping (1), after some algebra we find that the pulled back 2-form is given by

$$\begin{aligned}\omega_2 = 4\big[&\big(x_3^2 + x_4^2\big)\, dx_1 \wedge dx_2 \\ &+ (x_1x_4 - x_2x_3)(dx_1 \wedge dx_3 + dx_2 \wedge dx_4) \\ &- (x_1x_3 + x_2x_4)(dx_1 \wedge dx_4 - dx_2 \wedge dx_3) \\ &+ \big(x_1^2 + x_2^2\big)\, dx_3 \wedge dx_4\big].\end{aligned} \tag{4}$$

This expression can be further simplified by noticing $\sum_i x_i^2 = 1$ and $\sum_i x_i\, dx_i = 0$ successively. One can verify that

$$\begin{aligned}0 &= \Big(\sum_i x_i\, dx_i\Big)(x_1\, dx_2 - x_2\, dx_1 + x_3\, dx_4 - x_4\, dx_3) \\ &= \big[\big(x_1^2 + x_2^2\big)\, dx_1 \wedge dx_2 - (x_1x_4 - x_2x_3)(dx_1 \wedge dx_3 + dx_2 \wedge dx_4) \\ &\quad + (x_1x_3 + x_2x_4)(dx_1 \wedge dx_4 - dx_2 \wedge dx_3) \\ &\quad + \big(x_3^2 + x_4^2\big)\, dx_3 \wedge dx_4\big].\end{aligned} \tag{5}$$

Adding the above two equations together, we find that ω_2 can be rewritten as

$$\omega_2 = 4(dx_1 \wedge dx_2 + dx_3 \wedge dx_4). \tag{6}$$

Then it is easy to find that

$$\omega_1 = 2(x_1\, dx_2 - x_2\, dx_1 + x_3\, dx_4 - x_4\, dx_3). \tag{7}$$

Finally, the outer product of the above two differential forms gives the volume element of a unit S^3

$$\begin{aligned}\omega_1 \wedge \omega_2 = 8(&x_1\, dx_2 \wedge dx_3 \wedge dx_4 - x_2\, dx_1 \wedge dx_3 \wedge dx_4 \\ &+ x_3\, dx_1 \wedge dx_2 \wedge dx_4 - x_4\, dx_1 \wedge dx_2 \wedge dx_3).\end{aligned} \tag{8}$$

Since the surface area of the unit S^3 is $2\pi^2$, then the Hopf invariant of the map (1) is

$$\mathcal{H} = \frac{1}{16\pi^2} \int_{S^3} \omega_1 \wedge \omega_2 = 1. \tag{9}$$

For later use, the above statements can be elaborated in a more physical language. We can express the pulled back 2-form ω_2 as a $U(1)$ gauge field strength. Hence ω_1 becomes the corresponding gauge potential. Introducing a set of coordinate parameters $u_{1,2,3}$ to describe S^3, then the 2-form ω_2 can again be evaluated by pulling back Ω given by (3)

$$\omega_2 = \frac{1}{2} F_{\mu\nu}\, du_\mu \wedge du_\nu, \qquad \omega_1 = A_\mu\, du_\mu \quad \text{for } \mu, \nu = 1, 2, 3, \tag{10}$$

where in the component form $F_{\mu\nu}$ is the surface area element of S^2

$$F_{\mu\nu} = \epsilon^{ijk} y_i \partial_\mu y_j \partial_\nu y_k = -2i(\partial_\mu \bar{z}_i \partial_\nu z_i - \partial_\nu \bar{z}_i \partial_\mu z_i). \tag{11}$$

In the second equality, we have inserted the Hopf mapping and used the equality (A.2). We will visit this equality later in details. Moreover, the gauge potential A_μ can be easily found by $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

$$A_\mu = -i\big[\bar{z}_i(\partial_\mu z_i) - (\partial_\mu \bar{z}_i) z_i\big]. \tag{12}$$

In this language, the Hopf invariant can be expressed as a Chern–Simons term

$$\mathcal{H} = \frac{1}{32\pi^2} \int d^3u\, \epsilon_{\mu\nu\lambda} A_\mu F_{\nu\lambda} = 1. \tag{13}$$

3. Topological defect solution based on Hopf mapping

We consider a Yang–Mills–Higgs model in the 3+1D spacetime with the Lagrangian given as follows

$$L = -\frac{1}{2} D_\mu \phi^a \cdot \left(D_\mu \phi^a\right)^\dagger - \frac{1}{4} F^a_{\mu\nu} F^a_{\mu\nu} - V(\phi), \tag{14}$$

where ϕ^a is in the adjoint representation of an $SU(2)$ gauge group such that $D_\mu \phi^a = \partial_\mu \phi^a + e\epsilon^{abc} A^b_\mu \phi^c$, and $V(\phi) = \lambda(\phi^a\phi^a - v^2)^2$. We want to construct an instanton-type solution, the physical meaning of which will be clarified later. To achieve this, we consider the Euclidean version of the above model, which is equivalent to consider the spatial part of the 4+1D model. The spatial coordinates are chosen as x_i for $i = 1, 2, 3, 4$. We introduce $z_1 = x_1 + ix_2$ and $z_2 = x_3 + ix_4$ and define $r^2 = \sum_i |z_i|^2$. For simplicity, we set $v = 1$ and $e = 1$. As $r \to \infty$, ϕ^a approaches the classic vacuum solution as $\lim_{r\to\infty} \phi^a(\mathbf{x}) = m^a(\mathbf{x})$ with $m^a m^a = 1$ so that the potential $V(\phi)$ is minimized. Therefore, when $r \to \infty$, the vacuum solution is in fact a Hopf mapping which reads $S^3 \xrightarrow{m^a} S^2$. We can define the map m^a as $m^a(\mathbf{x}) = \frac{\bar{z}_i \sigma^a_{ij} z_j}{r^2}$. Now the $SU(2)$ gauge symmetry is spontaneously broken to the $U(1)$ symmetry.

To minimize the total energy of the instanton, one needs to require $D_\mu \phi^a = 0$ in the limit of large r. Next we multiply both sides of $D_\mu \phi^a = 0$ by $\epsilon^{ija} m^j$, and use the identity $\epsilon^{ija}\epsilon^{abc} = \delta^{ib}\delta^{jc} - \delta^{jb}\delta^{ic}$ to get

$$A^i_\mu = -\epsilon^{ija} m^j \partial_\mu m^a + A^b_\mu m^b m^i. \tag{15}$$

If we can find a solution to A^a_μ which is perpendicular to m^a in the space of vacua, i.e., $A^a_\mu m^a = 0$, then we get a simple expression of $A^i_\mu = -\epsilon^{ija} m^j \partial_\mu m^a$, i.e., A^a_μ is a large gauge transformation as $r \to \infty$. For convenience, we define the shorthand notation that $\partial_\mu = \partial_i, \bar{\partial}_i$ with $\partial_i = \frac{\partial}{\partial z_i}$ and $\bar{\partial}_i = \frac{\partial}{\partial \bar{z}_i}$. Hence the derivatives of ϕ^a at $r \to \infty$ are

$$\bar{\partial}_k m^a = \frac{\sigma^a_{kj} z_j}{r^2} - m^a \frac{z_k}{r^2}, \qquad \partial_k m^a = \frac{\sigma^a_{jk} \bar{z}_j}{r^2} - m^a \frac{\bar{z}_k}{r^2}. \tag{16}$$

By using the identity (A.2), one can further verify that

$$\begin{aligned} \bar{A}^a_k &= -i \frac{1}{r^4} \left(\sigma^a_{il} \delta_{jk} - \sigma^a_{kj} \delta_{il}\right) \bar{z}_i z_j z_l \\ &= -i \frac{1}{r^2} \left(m^a z_k - \sigma^a_{kj} z_j\right) \\ &= i \bar{\partial}_k m^a. \end{aligned} \tag{17}$$

By a similar calculation, we also get $A^a_k = -\epsilon^{abc} m^b \partial_k m^c = -i\partial_k m^a$. Since $m^a m^a = 1$, then $m^a \partial_\mu m^a = 0$, therefore the solution does satisfy $A^a_\mu m^a = 0$ and our derivation is indeed self-consistent. From above discussions we see that $\partial_\mu \phi^a$ approaches 0 as fast as $1/r$ when $r \to \infty$ while $D_\mu \phi^a$ vanishes identically.

Now we know the needed asymptotic behaviors of the fields ϕ^a and A^a_μ. To find the full defect solution we adopt the following assumption

$$\begin{aligned} &\phi^a(\mathbf{x}) = f(r) m^a(\mathbf{x}), \qquad A^a_k(\mathbf{x}) = -i\partial_k m^a(\mathbf{x}) g(r), \\ &\bar{A}^a_k(\mathbf{x}) = i\bar{\partial}_k m^a(\mathbf{x}) g(r), \end{aligned} \tag{18}$$

where the continuous functions f and g are required to satisfy $f(r) \to 1$, $g(r) \to 1$ as $r \to \infty$ and $f(0) = g(0) = 0$ (so that the scalar and gauge fields have well behaviors at the origin).

To evaluate the total action of the defect, we need to know the various derivatives of m^a, which are outlined in Appendix A. Using these results, it is straightforward to get the covariant derivatives

$$\begin{aligned} \bar{D}_k \phi^a &= \bar{\partial}_k m^a f + m^a \frac{z_k}{2r} f' + i\epsilon^{abc} \bar{\partial}_k m^b m^c g f \\ &= \bar{\partial}_k m^a f(1-g) + m^a \frac{z_k}{2r} f', \\ D_k \phi^a &= \partial_k m^a f(1-g) + m^a \frac{\bar{z}_k}{2r} f', \end{aligned} \tag{19}$$

and the first term of the Lagrangian is given by

$$D_\mu \phi^a \left(D_\mu \phi^a\right)^\dagger = 4\bar{D}_k \phi^a D_k \phi^a = \frac{8f^2(1-g)^2}{r^2} + \left(f'\right)^2. \tag{20}$$

Using Eq. (A.9), the field strength is evaluated as

$$\begin{aligned} F^a_{i\bar{j}} &= \partial_i \bar{A}^a_j - \bar{\partial}_j A^a_i + \epsilon^{abc} A^b_i \bar{A}^c_j \\ &= 2i\partial_i \bar{\partial}_j m^a g + \epsilon^{abc} \partial_i m^b \bar{\partial}_j m^c g^2 + \frac{i\bar{z}_i \bar{\partial}_j m^a}{2r} g' + \frac{iz_j \partial_i m^a}{2r} g' \\ &= \frac{2i(r^2\delta_{ij} - \bar{z}_i z_j)}{r^4} m^a (g^2 - 2g) + \left(\frac{i\bar{z}_i \bar{\partial}_j m^a}{2r} + \frac{iz_j \partial_i m^a}{2r}\right) g'. \end{aligned} \tag{21}$$

To simplify our notation, we define the coefficient of g' as

$$C^a_{i\bar{j}} \equiv \frac{i\bar{z}_i \bar{\partial}_j m^a}{2r} + \frac{iz_j \partial_i m^a}{2r} = i\frac{\bar{z}_i \sigma^a_{jk} z_k + \bar{z}_k \sigma^a_{ki} z_j - 2m^a \bar{z}_i z_j}{2r^3}. \tag{22}$$

It is easy to see that $C^a_{i\bar{i}} = 0$ and $m^a C^a_{i\bar{j}} = 0$. Similarly, by making use of Eq. (A.10), the other component of the field strength with two holomorphic or two anti-holomorphic indices are given by

$$\begin{aligned} F^a_{ij} &= -i\left(\frac{\bar{z}_i \partial_j m^a}{2r} - \frac{\bar{z}_j \partial_i m^a}{2r}\right) g', \\ F^a_{\bar{i}\bar{j}} &= i\left(\frac{z_i \bar{\partial}_j m^a}{2r} - \frac{z_j \bar{\partial}_i m^a}{2r}\right) g'. \end{aligned} \tag{23}$$

The expressions of all the field strength in the complex indices are list in Appendix A. The squares of the field strength can be simplified by noting the following relations between the field strengths with complex and real indices

$$\begin{aligned} F^a_{z_1,z_2} &= \frac{1}{4}\left(F^a_{13} - F^a_{24} - iF^a_{14} - iF^a_{23}\right), \\ F^a_{z_1,\bar{z}_2} &= \frac{1}{4}\left(F^a_{13} + F^a_{24} + iF^a_{14} - iF^a_{23}\right), \\ F^a_{z_1,\bar{z}_1} &= \frac{i}{2} F^a_{12}, \qquad F^a_{z_2,\bar{z}_2} = \frac{i}{2} F^a_{34}. \end{aligned} \tag{24}$$

From these relations, we find

$$\sum_{\mu,\nu} F^2_{\mu\nu} = 8\sum_{i,j} (F_{z_i,\bar{z}_j} F_{\bar{z}_i,z_j} + F_{\bar{z}_i,\bar{z}_j} F_{z_i,z_j}). \tag{25}$$

Since $m^a C^a_{i\bar{j}} = 0$, the cross term vanishes in the square of $F^a_{i,\bar{j}}$. Using the fact that $z_i \partial_i m^a = \bar{z}_i \bar{\partial}_i m^a = 0$, it is easy to find that

$$\left|\frac{2i(r^2\delta_{ij} - \bar{z}_i z_j)}{r^4}\right|^2 = \frac{4}{r^4}, \qquad C^a_{\bar{i}j} C^a_{i\bar{j}} = \frac{1}{r^2}. \tag{26}$$

Thus we find

$$F^a_{\bar{i}j} F^a_{i\bar{j}} = \frac{4(2g - g^2)^2}{r^4} + \frac{(g')^2}{r^2}. \tag{27}$$

Similarly, we have

$$\begin{aligned} F^a_{\bar{i}\bar{j}} F^a_{i,j} &= \frac{1}{4r^2}\left(z_i \bar{\partial}_j m^a - z_j \bar{\partial}_i m^a\right)\left(\bar{z}_i \partial_j m^a - \bar{z}_j \partial_i m^a\right)\left(g'\right)^2 \\ &= \frac{(g')^2}{r^2}. \end{aligned} \tag{28}$$

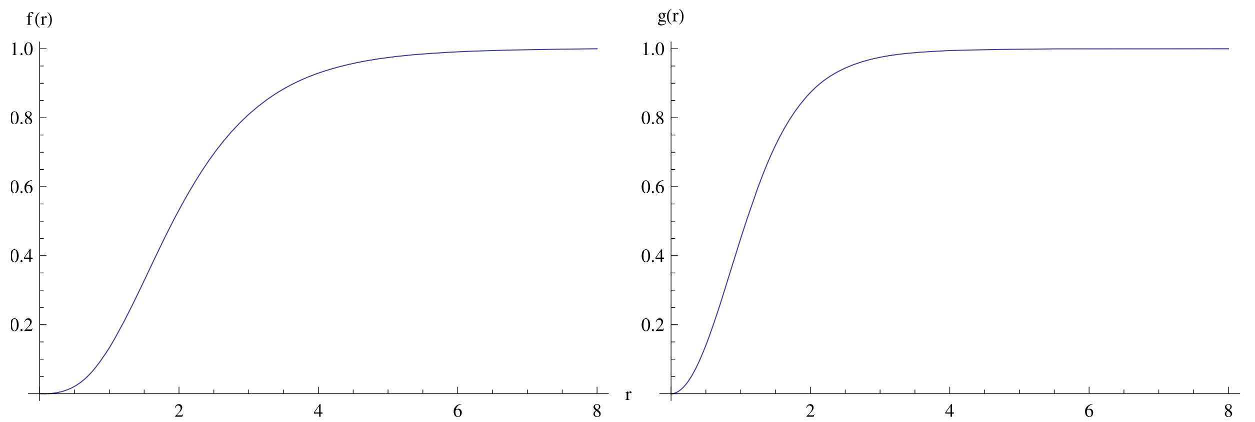

Fig. 1. Numerical results of functions $f(r)$ and $g(r)$ vs. r.

Collecting all the above results, we finally get the total energy (or total action in the Euclidean space) as

$$S = \int r^3 dr \left[\frac{4f^2(1-g)^2}{r^2} + \frac{(f')^2}{2} + 4\left(\frac{2(2g-g^2)^2}{r^4} + \frac{(g')^2}{r^2} \right) + \lambda \left(f^2 - 1\right)^2 \right]. \tag{29}$$

This result is quite similar to the magnetic monopole energy. The difference is that here we have to integrate the whole 3-sphere. Therefore the action is actually logarithm divergence. This situation is very similar to the vortex solution of scalar $O(2)$ model without coupling any gauge field in the two-dimensional case. Therefore this type of soliton or instanton may not contribute in a infinitely large system. To get some physical effect, we should consider a finite sized system such that the radial integral has an upper bound. Then it makes sense to minimize the total action to find the instanton solutions.

As usual, we use the variational method to find that f, g satisfy the following equations

$$\begin{aligned} &\frac{d^2 g}{dr^2} + \frac{1}{r}\frac{dg}{dr} + f^2(1-g) - \frac{4(1-g)(2g-g^2)}{r^2} = 0, \\ &\frac{d^2 f}{dr^2} + \frac{3}{r}\frac{df}{dr} - \frac{8f(1-g)^2}{r^2} - 4\lambda f\left(f^2-1\right) = 0 \end{aligned} \tag{30}$$

with the boundary conditions $g(\infty) = f(\infty) = 1$ and $f(0) = g(0) = 0$. These two equations are coupled nonlinear differential equations, which in general cannot be solved analytically. Here we numerically solve the above differential equations. For convenience, we have taken $\lambda = 1$. The numerical results are shown in Fig. 1. Here we use a large fixed upper bound r_c to replace the infinity.

4. Topological charge

For the 't Hooft–Polyakov magnetic monopole, the magnetic charge is also the winding number of the mapping between the spatial boundary and the vacuum manifold. Therefore, the magnetic charge is also the topological charge, and it is quantized naturally. For our defect solution, the topological charge is obviously related to the Hopf invariant. But its physical meaning is not as intuitive as that of the magnetic charge. At the boundary of the spacetime, the $SU(2)$ gauge symmetry breaks down to the $U(1)$ gauge symmetry as we discussed in Section 3. For magnetic monopoles, it is this $U(1)$ gauge field that gives rise to a hedgehog-like magnetic field configuration. For our defect solution, we identify that the A_μ field we introduced in the end of Section 2 is exactly this unbroken $U(1)$ gauge field.

We first give a warm-up discussion on the monopole. At the boundary, A_μ^a is designed to cancel $\partial_\mu \phi^a$. The most possible form of A_μ^a that satisfies this requirement is

$$A_\mu^a = \epsilon_{abc}\phi^b \partial_\mu \phi^c + \phi^a A_\mu^{(1)}. \tag{31}$$

Here $A_\mu^{(1)} = A_\mu^a \phi^a$ is the gauge field associated with the unbroken $U(1)$ symmetry. Then the corresponding field strength is

$$F_{\mu\nu}^a = \phi^a \mathcal{F}_{\mu\nu}, \quad \mathcal{F}_{\mu\nu} = \partial_\mu A_\nu^{(1)} - \partial_\nu A_\mu^{(1)} + \epsilon_{abc}\phi^a \partial_\mu \phi^b \partial_\nu \phi^c. \tag{32}$$

However, the relation between the field strength and the associated vector potential is not simply $\mathcal{F}_{\mu\nu} = \partial_\mu A_\nu^{(1)} - \partial_\nu A_\mu^{(1)}$ as in the situation of the usual $U(1)$ symmetry. The extra term $\epsilon_{abc}\phi^a \partial_\mu \phi^b \partial_\nu \phi^c$ in Eq. (32) cannot be written as the form $\partial_\mu A_\nu - \partial_\nu A_\mu$ with some vector potential A_μ. In the mathematical language, this term is closed but not exact. It is precisely this term that is responsible for the monopole-like field configuration and makes the crucial contribution to the topological charge. The magnetic charge of the monopole is given by the integration of $\mathcal{F}_{\mu\nu}$ over the spatial boundary

$$g = \int dS_{\mu\nu}\, \mathcal{F}_{\mu\nu} = \int dS_{\mu\nu}\, \epsilon_{abc}\phi^a \partial_\mu \phi^b \partial_\nu \phi^c. \tag{33}$$

Clearly, the first term of $\mathcal{F}_{\mu\nu}$ does not make any contribution. The second term is the winding number of the vacuum configuration mapping as we mentioned before.

On the contrary, for our Hopf defect solution, the extra term in Eq. (32) can be expressed as a curl of a vector potential due to the pulling back of the Hopf mapping, as we discussed in Section 2. Thus it can be expressed as

$$\epsilon_{abc}\phi^a \partial_\mu \phi^b \partial_\nu \phi^c = \partial_\mu A_\nu^{(2)} - \partial_\nu A_\mu^{(2)}. \tag{34}$$

Then the field strength associated with the unbroken $U(1)$ symmetry is

$$\mathcal{F}_{\mu\nu} = \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu, \quad \mathcal{A}_\mu = A_\mu^{(1)} + A_\mu^{(2)}. \tag{35}$$

In terms of $\mathcal{F}_{\mu\nu}$ and $\mathcal{A}_\mu$, we can construct the Hopf invariant as in Section 2

$$\mathcal{H} = \frac{1}{32\pi^2} \int d^3x\, \epsilon_{\mu\nu\lambda} \mathcal{A}_\mu \mathcal{F}_{\nu\lambda}. \tag{36}$$

Here the integral is over the three-dimensional boundary of the 4D spacetime. Since $A_\mu^{(1)}$ is topologically trivial, it is easy to see that terms like $\epsilon_{\mu\nu\lambda}A_\mu^{(1)}\partial_\nu A_\lambda^{(1)}$, $\epsilon_{\mu\nu\lambda}A_\mu^{(1)}\partial_\nu A_\lambda^{(2)}$ and $\epsilon_{\mu\nu\lambda}A_\mu^{(2)}\partial_\nu A_\lambda^{(1)}$ do not contribute to the above integral. By the construction of the Hopf defect solution, we have

$$\epsilon_{abc}\phi^a\partial_\mu\phi^b\partial_\nu\phi^c\big|_{r\to\infty}=\epsilon_{abc}m^a\partial_\mu m^b\partial_\nu m^c=\partial_\mu A_\nu^{(2)}-\partial_\nu A_\mu^{(2)} \tag{37}$$

with $A_\mu^{(2)}=-i[\bar{\zeta}_i(\partial_\mu\zeta_i)-(\partial_\mu\bar{\zeta}_i)\zeta_i]$ and $\zeta_i=z_i/r$. Therefore, we find

$$\mathcal{H}=\frac{1}{16\pi^2}\int d^3x\,\epsilon_{\mu\nu\lambda}A_\mu^{(2)}\partial_\nu A_\lambda^{(2)}=1 \tag{38}$$

according to Eq. (13). This result can also be directly obtained from $\mathcal{F}_{\mu\nu}$ at the boundary. All components of $\mathcal{F}_{\mu\nu}$ can be obtained from the expressions in Appendix A by noting $g(\infty)=1$ and $g'(\infty)=0$

$$\begin{aligned}&\mathcal{F}_{12}=\frac{4(x_3^2+x_4^2)}{r^4}, &&\mathcal{F}_{13}=\mathcal{F}_{24}=\frac{4(x_1x_4-x_2x_3)}{r^4},\\ &\mathcal{F}_{34}=\frac{4(x_1^2+x_2^2)}{r^4}, &&\mathcal{F}_{14}=-\mathcal{F}_{23}=-\frac{4(x_1x_3+x_2x_4)}{r^4}.\end{aligned} \tag{39}$$

If we express $\mathcal{F}_{\mu\nu}$ as a differential form, we find

$$\mathcal{F}=\frac{1}{2}\mathcal{F}_{\mu\nu}\,dx_\mu\wedge dx_\nu=-\omega_2. \tag{40}$$

Here ω_2 is the pulled back volume element form as we defined in Section 2. Follow the same steps, we find

$$\mathcal{H}=\frac{1}{16\pi^2}\int\mathcal{A}\wedge\mathcal{F}=\frac{1}{16\pi^2}\int\omega_1\wedge\omega_2=1. \tag{41}$$

5. Qualitative picture of Hopf defect

To find a qualitative picture of Hopf defect, we must have a better understanding about the geometric meaning of the Hopf mapping. Since it is very difficult to visualize a 3-sphere embedded inside $\mathbb{R}^4$, we consider a deformed and simplified version of the Hopf mapping. We first deform S^3 into a cylinder $S^2\times[0,1]$, then treat S^2 as a one-point-compactification of $\mathbb{R}^2$. Now we show the following map: $\mathbb{R}^2\times[0,1]\to S^2$ has a nontrivial Hopf number

$$\begin{aligned}y_1&=\frac{1}{r}\sin\big[f(r)\big]\big(x_1\cos\big[a(x_3)\big]-x_2\sin\big[a(x_3)\big]\big),\\ y_2&=\frac{1}{r}\sin\big[f(r)\big]\big(x_1\sin\big[a(x_3)\big]+x_2\cos\big[a(x_3)\big]\big),\\ y_3&=\cos\big[f(r)\big].\end{aligned} \tag{42}$$

Here $(x_1,x_2)\in\mathbb{R}^2$, $x_3\in[0,1]$ and $r=\sqrt{x_1^2+x_2^2}$. We also assume that $a(x_3)$ is a monotonic function from $[0,1]$ to $[0,2\pi]$ and $f(0)=\pi$, $f(\infty)=0$.

If we treat x_3 as a time variable, the above mapping describes that $\mathbb{R}^2$ makes a 2π rotation when time evolves from 0 to 1. Hence the world line of each point in $\mathbb{R}^2$ produces a helix curve. If we identify the initial time with the final time, the world line becomes a closed loop. Moreover, the world lines of two different points become two linked loops. In this sense, we expect that this map can give the similar result as the Hopf mapping. But we should mention that the map $S^2\times S^1\to S^2$ is topologically quite different from $S^3\to S^2$. For $S^2\times S^1\to S^2$, the actual topological invariants are the winding number of $S^2\to S^2$ and the twisting number along the S^1 [17]. In this case, the Hopf invariant is only an approximate topological invariant. We will only use $S^2\times S^1\to S^2$ to help us to visualize the Hopf mapping.

The volume element or the field strength is determined by $F_{\mu\nu}=\epsilon^{ijk}y_i\partial_\mu y_j\partial_\nu y_k$. Thus we find

$$\begin{aligned}&F_{12}=\frac{1}{r}\sin\big[f(r)\big]f'(r), \qquad F_{23}=\frac{x_2}{r}\sin\big[f(r)\big]f'(r)a'(x_3),\\ &F_{31}=-\frac{x_1}{r}\sin\big[f(r)\big]f'(r)a'(x_3).\end{aligned} \tag{43}$$

The corresponding vector potentials are

$$\begin{aligned}&A_1=\frac{x_2}{r}\cos\big[f(r)\big], \qquad A_2=-\frac{x_1}{r}\cos\big[f(r)\big],\\ &A_3=-a'(x_3)\cos\big[f(r)\big].\end{aligned} \tag{44}$$

Finally, we find

$$\begin{aligned}\mathcal{H}&=\frac{1}{16\pi^2}\int d^3x\,\epsilon_{\mu\nu\lambda}A_\mu F_{\nu\lambda}\\ &=\frac{1}{8\pi^2}\int d^3x\,\frac{1}{r}\sin\big[f(r)\big]f'(r)a'(x_3)=1\end{aligned} \tag{45}$$

This result reflects that the linking number of two world lines is 1 just as that of the Hopf mapping.

Based on the above discussion, we can have the following geometric picture of Hopf mapping. Recall that S^3 can be decomposed into two solid tori. In complex coordinates, S^3 is described by $|z_1|^2+|z_2|^2=1$. Then the two solid tori are

$$T_1:\quad 1/2<|z_1|^2<1, \qquad |z_2|^2=1-|z_1|^2; \tag{46}$$

$$T_2:\quad 0<|z_1|^2<1/2, \qquad |z_2|^2=1-|z_1|^2. \tag{47}$$

There are two types of nontrivial cycles on the torus which are also the generators of the $\pi_1(T^2)$. These two tori T_1 and T_2 are related by a modular transformation which exchanges the two types of cycles. If the torus is characterized by a complex number τ, then this modular transformation is given by $\tau\to-\frac{1}{\tau}$. It is easy to verify that, under the Hopf mapping, the pre-image of the northern hemisphere S^N is just T_1 and that of the southern hemisphere S^S is T_2.

It can be found that the pre-image of a fixed point on S^N is a circle described by $(z_1e^{i\phi},z_2e^{i\phi})$. Here $e^{i\phi}$ is an arbitrary phase factor and $z_{1,2}$ are fixed complex numbers solved from the Hopf mapping equations. If we trace the trajectory of the vector $(z_1e^{i\phi},z_2e^{i\phi})$, we find that it makes a 2π rotation on the z_1 plane, and a 2π rotation on the z_2 plane simultaneously. The resulting curve is a helix with the starting and ending points identified. Since a solid torus can be treated as $D^2\times S^1$ (D^2 is a 2D disc), the pre-image of each point on S^N is a point on D^2 which makes a 2π rotation as we travel along S^1. This is also true for the southern part S^S. After making a modular transformation, we can glue the southern part back to the northern part to get a map like $S^2\times S^1\to S^2$. If we cut S^1 into a interval $[0,1]$, we retrieve the map $S^2\times[0,1]\to S^2$ of Eq. (42).

With the above discussions, we may find a qualitative picture of the Hopf defect as follows. The Hopf defect a tunneling event such that the monopole makes a 2π rotation of the vacuum manifold S^2 along a closed monopole world line. In spacetime, the closed world line describes that a monopole and an anti-monopole are created at one point and then annihilated at some other place. The Hopf defect will make a $\pm\pi$ rotation of the vacuum manifold of the monopole and anti-monopole respectively. In a single monopole solution, the vacuum manifold and spatial boundary are locked and the rotation of vacuum manifold is equivalent to the rotation of the spatial boundary. But in the case of Hopf defect, they are not equivalent because of the closed monopole line. This situation is very similar to the closed skyrmion string as discussed in [18].

Furthermore, the closed monopole line with twisted $U(1)$ modulus may lead to a Yang–Mills instanton, as suggested in [19–21].

If there is no Hopf number dependent term in the Hamiltonian, this tunneling event still has no direct physical effects. We know that the Hopf number term (36) is expressed as the surface integral of the boundary of the 4D spacetime. One might guess that the corresponding term in the bulk will be the θ vacuum term

$$\mathcal{H}=\frac{\theta}{32\pi^2}\int d^4x\,\epsilon_{\mu\nu\lambda\rho}F^a_{\mu\nu}F^a_{\lambda\rho}, \tag{48}$$

where θ is an arbitrary angle, since $\mathcal{F}$ is the unbroken component of the F^a at the spacetime boundary. However, a direct evaluation shows that the term $F^a\wedge F^a$ vanishes identically. This means that $\mathcal{F}$ cannot be simply replaced by F^a. Therefore, the correct term is

$$\begin{aligned}\mathcal{H}&=\frac{\theta}{32\pi^2}\int d^4x\,\epsilon_{\mu\nu\lambda\rho}\mathcal{F}_{\mu\nu}\mathcal{F}_{\lambda\rho}\\&=\frac{\theta}{32\pi^2}\int d^4x\,\epsilon_{\mu\nu\lambda\rho}\mathcal{F}^a_{\mu\nu}\mathcal{F}^b_{\lambda\rho}\phi^a\phi^b.\end{aligned} \tag{49}$$

This term is a higher order term which usually does not appear in ordinary gauge theories. But it may possibly appear in some low energy effective theory. If such terms appear, the monopole will pick up a extra phase factor when making a 2π rotation of the vacuum manifold.

6. Conclusion and discussion

We have constructed the Hopf defect solution in the 3+1D non-Abelian gauge theory based on a nontrivial Hopf mapping. The topological charge is identified with the Chern–Simons term of the unbroken $U(1)$ gauge field, which corresponds to some higher order term of the non-Abelian gauge field. The Hopf defect is a spacetime event that executes a 2π rotation of vacuum manifold of the monopole. The appearing of the Hopf defect and Hopf term together may generate a extra phase factor for the monopole under vacuum manifold rotation.

Since monopoles has not been discovered in nature yet, all the above discussions may seem to be of purely academic interests. However, non-Abelian monopoles can be realized in certain condensed matter systems such as superfluid A phase of ^{3}He [22]. Thus, in a finite system of ^{3}He, it is quite possible that Hopf defect may have real physical effects if the Hopf term appears in the low energy theory.

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Appendix A. Some useful formulas

In the main text, we make frequent use of the following identities of Pauli matrices

$$\sigma^a_{ij}\sigma^a_{kl}=2\delta_{il}\delta_{jk}-\delta_{ij}\delta_{kl}, \tag{A.1}$$

$$\epsilon^{abc}\sigma^b_{ij}\sigma^c_{kl}=i\big(\sigma^a_{il}\delta_{jk}-\sigma^a_{kj}\delta_{il}\big). \tag{A.2}$$

In computing the covariant derivative and field strength, we need to calculate the second order derivatives of m^a. When analyzing the asymptotic behavior of A^a_μ, we have got the first order derivatives of m^a

$$\partial_i m^a=-i\epsilon^{abc}m^b\partial_i m^c,\qquad \bar\partial_i m^a=i\epsilon^{abc}m^b\bar\partial_i m^c. \tag{A.3}$$

From the identity $m^a\partial_\mu m^a=0$, we can directly find the following results

$$z_i=m^a\sigma^a_{ij}z_j,\qquad \bar z_i=\bar z_j m^a\sigma^a_{ji}. \tag{A.4}$$

When evaluating the term $D_\mu\phi^a(D_\mu\phi^a)^\dagger$, we need to calculate the product of two first order derivatives of m^a. By using Eqs. (16), (A.1) and (A.4), we have

$$\begin{aligned}\partial_i m^a\partial_j m^a&=\frac{1}{r^4}\big(\sigma^a_{pi}\sigma^a_{qj}\bar z_p\bar z_q-m^a\sigma^a_{pi}\bar z_p\bar z_j-m^a\sigma^a_{qj}\bar z_i\bar z_q+\bar z_i\bar z_j\big)\\&=\frac{1}{r^4}\big[(2\delta_{pj}\delta_{iq}-\delta_{pi}\delta_{qj})\bar z_p\bar z_q-\bar z_i\bar z_j\big]\\&=0.\end{aligned} \tag{A.5}$$

Similarly, we can get $\bar\partial_i m^a\bar\partial_j m^a=0$. These results obviously imply $\partial_i m^a\partial_i m^a=\bar\partial_i m^a\bar\partial_i m^a=0$ immediately. Another type of product is evaluated similarly as

$$\begin{aligned}\partial_i m^a\bar\partial_j m^a&=\frac{1}{r^4}\big[(2\delta_{pq}\delta_{ij}-\delta_{pi}\delta_{jq})\bar z_p z_q-\bar z_i z_j\big]\\&=\frac{2(r^2\delta_{ij}-\bar z_i z_j)}{r^4}.\end{aligned} \tag{A.6}$$

This immediately implies $\partial_i m^a\bar\partial_i m^a=\frac{2}{r^2}$.

When computing the term $F^a_{\mu\nu}F^a_{\mu\nu}$, we must know the quantities like $\partial_\mu\partial_\nu m^a$ and $\epsilon^{abc}m^a\partial_\mu m^b\partial_\nu m^c$. We don't need to worry about $\partial_i\partial_j m^a$ since they are cancelled in the expressions of F^a_{ij}. To evaluate $\partial_i\bar\partial_j m^a$, we first start from the expression (16) and take one more derivative

$$\partial_i\bar\partial_j m^a=\frac{\sigma^a_{ji}}{r^2}-\frac{\sigma^a_{jk}z_k\bar z_i}{r^4}-\frac{\sigma^a_{ki}z_j\bar z_k}{r^4}+2m^a\frac{\bar z_i z_j}{r^4}. \tag{A.7}$$

We further have

$$\begin{aligned}\epsilon^{abc}m^b\partial_i\bar\partial_j m^c&=\epsilon^{abc}\bigg(\frac{\sigma^b_{pq}\sigma^c_{ji}\bar z_p z_q}{r^4}-\frac{\sigma^b_{pq}\sigma^c_{jk}\bar z_p z_q z_k\bar z_i}{r^6}\\&\qquad-\frac{\sigma^b_{pq}\sigma^c_{ki}\bar z_p z_q z_j\bar z_k}{r^6}+2m^b m^c\frac{\bar z_i z_j}{r^4}\bigg)\\&=0,\end{aligned} \tag{A.8}$$

where the properties $\epsilon^{abc}m^b m^c=0$, $\bar z_i z_i=r^2$ and Eq. (A.2) have been applied. We can also start from the second identity of Eqs. (A.3) to calculate the second order derivative of m^a

$$\begin{aligned}\partial_i\bar\partial_j m^a&=i\epsilon^{abc}\partial_i m^b\bar\partial_j m^c\\&=\epsilon^{abc}\epsilon^{bpq}m^p\partial_i m^q\bar\partial_j m^c\\&=\big(\delta^{aq}\delta^{cp}-\delta^{ap}\delta^{cq}\big)m^p\partial_i m^q\bar\partial_j m^c\\&=-\frac{2(r^2\delta_{ij}-\bar z_i z_j)}{r^4}m^a,\end{aligned} \tag{A.9}$$

where Eq. (A.8) has been applied in the second line, $m^a\partial_\mu m^a=0$ has been applied in the last line. To determine $\epsilon^{abc}m^a\partial_i m^b\partial_j m^c$, we use Eq. (A.3) again

$$\begin{aligned}\epsilon^{abc}m^a\partial_i m^b\partial_j m^c&=-im^a\epsilon^{abc}\epsilon^{bpq}m^p\partial_i m^q\partial_j m^c\\&=im^a\big(m^a\partial_i m^c\partial_j m^c-m^c\partial_i m^a\partial_j m^c\big)\\&=0,\end{aligned} \tag{A.10}$$

where the equality (A.5) has been applied. Similarly, we can show that $\epsilon^{abc}m^a\bar{\partial}_i m^b\bar{\partial}_j m^c = 0$.

For completeness, we list all the components of field strength as follows

$$F^a_{12} = \frac{\bar{z}_i(\sigma^2\sigma^a)_{ij}\bar{z}_j}{2r^3}g', \tag{A.11}$$

$$F^a_{\bar{1}\bar{2}} = \frac{z_i(\sigma^a\sigma^2)_{ij}z_j}{2r^3}g', \tag{A.12}$$

$$F^a_{1\bar{1}} = \frac{2i|z_2|^2m^a(2g-g^2)}{r^4} + i\frac{r^4\delta_{a3} - (\bar{z}_i\sigma^a_{ij}z_j)(\bar{z}_i\sigma^3_{ij}z_j)}{2r^5}g', \tag{A.13}$$

$$F^a_{1\bar{2}} = \frac{-2i\bar{z}_1z_2m^a(2g-g^2)}{r^4} + i\frac{r^4\delta^a - 2\bar{z}_1z_2(\bar{z}_i\sigma^a_{ij}z_j)}{r^5}g', \tag{A.14}$$

$$F^a_{2\bar{1}} = \frac{-2i\bar{z}_2z_1m^a(2g-g^2)}{r^4} + i\frac{r^4\bar{\delta}^a - 2\bar{z}_2z_1(\bar{z}_i\sigma^a_{ij}z_j)}{r^5}g', \tag{A.15}$$

$$F^a_{2\bar{2}} = \frac{2i|z_1|^2m^a(2g-g^2)}{r^4} - i\frac{r^4\delta_{a3} - (\bar{z}_i\sigma^a_{ij}z_j)(\bar{z}_i\sigma^3_{ij}z_j)}{2r^5}g'. \tag{A.16}$$

Here we define $\delta^a = (1, i, 0)$ and $\bar{\delta}^a = (1, -i, 0)$.

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